Quadratic Formula

If $ax^2+bx+c=0$, what values of *x* comprise the solution?

First, divide through by *a*:

$$\frac{ax^2}{a}+\frac{bx}{a}+\frac{c}{a}=x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Add and subtract $\left(\frac{b}{2a}\right)^2$, so that we have

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2+\frac{c}{a}-\left(\frac{b}{2a}\right)^2=0$$

Now, note that the first three terms can be rewritten as follows:

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(x+\frac{b}{2a}\right)^2$$

and, therefore,

$$\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\left(\frac{b}{2a}\right)^2=0$$

or

$$\left(x+\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

Now, taking the square root of each side of this equation we have

$$x+\frac{b}{2a}=\pm\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}}=\pm\sqrt{\left(\frac{b}{2a}\right)^2-\left(\frac{2a}{2a}\right)^2\frac{c}{a}}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

Therefore,

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

which is the well-known quadratic formula, usually written in the form:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$